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An LGSM to identify nonhomogeneous heat conductivity functions by an extra measurement of temperature

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Abstract

Considering here is an inverse problem for estimating the unknown nonhomogeneous heat conductivity function $\alpha(x)$ in $T_t(x,t) = \partial(\alpha(x)T_x)/\partial x + h(x,t)$ with the aid of an extra measurement of temperatures at a final time, which may be disturbed by random noise. A Lie-group shooting method (LGSM) is developed from the one-step Lie-group elements obtained by a spatial-discretization of heat conduction equation and by using the central difference or forward difference for $\alpha'(x)$ in spatial domain. The heat conductivities are available by directly solving linear equations. The new methods have twofold advantages in that no a priori information of heat conductivity is required and no iterations in the calculation process are needed. The accuracy and robustness of present methods are confirmed by comparing estimated results with exact solutions. The LGSM is stable and accurate, although the estimations are carried out under a large measurement noise.

Keywords: Inverse problem; Parameter identification; Lie-group shooting method; Nonhomogeneous heat conductivity; Lie-group method

1. Introduction

Present study aims to estimate as accurately as possible the nonhomogeneous heat conductivity parameter by solving an inverse heat conduction problem under an overspecified final time temperature, which can be acquired through measurements by thermocouples in a heat conducting rod. This identification problem can find wide range applications in engineering and science. For new materials, it is often easier to measure the temperatures at some points in the medium at a certain time, rather than that to directly measure the thermophysical parameters themselves. Due to its importance on the knowledge of thermophysical properties for new materials used in many thermal system analyses, this inverse problem has already attracted much attentions.

The parameters determination in partial differential equations (PDEs) from overspecified data are widely encountered in the modeling of physical phenomena. We consider an inverse problem of finding an unknown parameter $\alpha(x)$ in a one-dimensional heat conduction equation, of which one needs to find the temperature distribution $T(x,t)$ as well as the heat conductivity function $\alpha(x)$ that simultaneously satisfy

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\alpha(x)\frac{\partial T(x,t)}{\partial x}\right] + h(x,t),$$
$$0 < x < \ell, \ 0 < t \leqslant t_f, \tag{1}$$

$$T(0,t) = F_0(t), \qquad T(\ell,t) = F_\ell(t), \tag{2}$$

$$T(x,0) = f(x), \tag{3}$$

where $h(x,t)$ is a heat generating function, and $F_0(t)$, $F_\ell(t)$ and $f(x)$ are respectively the given functions of left-boundary temperature, right-boundary temperature and initial temperature.

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Nomenclature

$\mathbf{A}$	augmented matrix
a, b	coefficients defined in Eqs. (28), (31) and (37)
a_i	coefficient defined in Eq. (56)
b_i	coefficient defined in Eq. (57)
c_i	coefficient defined in Eq. (58)
$\mathbf{B}$	matrix defined by Eq. (59)
C	integration constant
$\mathbf{c}$	vector defined by Eq. (59)
$\mathbf{d}$	vector defined in Eq. (63)
$\mathbf{D}$	matrix defined in Eq. (62)
e_i	normally distributed random error
$\mathbf{f}$	n-dimensional vector field
$\hat{\mathbf{f}}$	$:= \mathbf{f}(\hat{t}, \widehat{\mathbf{T}})$
$\hat{f}_i$	the ith component of $\hat{\mathbf{f}}$
$f(x)$	initial temperature function
$F_0(t)$	left-boundary function
$F_\ell(t)$	right-boundary function
$F_m(x)$	temperature function at final time t_f
$\mathbf{F}$	$:= \hat{\mathbf{f}}/\|\widehat{\mathbf{T}}\|$
$\mathbf{g}$	$(n+1)$-dimensional Minkowski metric
$\mathbf{G}$	an element of Lorentz group
$\mathbf{G}_i,\ i = 1, \ldots, K$	elements of Lorentz group
$\mathbf{G}(r)$	an element of Lorentz group
$\mathbf{G}(t_f)$	an element of Lorentz group
G_0^0	the 00-th component of $\mathbf{G}$
$h(x, t)$	heat generating function
$h_i(t)$	$:= h(x_i, t)$
$\hat{h}_i$	$:= h_i(\hat{t})$
$\mathbf{I}_n$	n-dimensional unit matrix
ℓ	length of rod
$\|\bullet\|$	Euclidean norm
$\mathbb{M}^{n+1}$	$(n+1)$-dimensional Minkowski space
n	number of interior grid points
r	weighting factor
S	$:= t_f\|\mathbf{T}^f - \mathbf{T}^0\|$
$SO_o(n, 1)$	$(n+1)$-dimensional Lorentz group
$so(n, 1)$	the Lie algebra of $SO_o(n, 1)$
t	time
t_f	final time
t_0	$:= t_f/2$
$\hat{t}$	$:= (1 - r)t_f$
Δt	time stepsize
T	temperature
$\mathbf{T}$	temperature vector of T_i
$\mathbf{T}^0$	initial temperature vector
$\mathbf{T}^f$	temperature vector at final time t_f
$\widehat{\mathbf{T}}$	$:= r\mathbf{T}^0 + (1 - r)\mathbf{T}^f$
$T_i(t)$	$:= T(x_i, t)$
$\widehat{T}_i$	the ith component of $\widehat{\mathbf{T}}$
x	space variable
x_i	discretized coordinate of x
Δx	mesh size of x
$\mathbf{X}$	$(n+1)$-dimensional augmented vector
$\mathbf{X}_k$	numerical value of $\mathbf{X}$ at the kth time step
$\mathbf{X}^0$	the value of $\mathbf{X}$ at initial time
$\mathbf{X}^f$	the value of $\mathbf{X}$ at final time t_f
Z	$:= \exp(S/\eta)$

Greek symbols

$\alpha(x)$	spatial-dependent heat conductivity
α_i	$:= \alpha(x_i)$
$\boldsymbol{\alpha}$	vector form of α_i
η	coefficient defined in Eq. (34)
θ	intersection angle of $\mathbf{T}^f - \mathbf{T}^0$ and $\mathbf{T}^0$
σ	standard deviation of measurement errors

Subscripts and superscripts

i	index
K	index
t	transpose

Because the above problem has an unknown function $\alpha(x)$, it cannot be solved directly to find $T(x, t)$. This point is drastically different from the direct problem, where $\alpha(x)$ is given, and the solution of $T(x, t)$ is available by using general numerical methods. Here, ℓ is a length of the heat conducting rod, and t_f is a terminal time, at which an over-specified final temperature

$$T(x, t_f) = F_m(x) \tag{4}$$

is required, in order to have enough information to estimate the unknown function $\alpha(x)$.

We will develop a new Lie-group shooting method (LGSM) for the inverse problem of parameter identification governed by Eqs. (1)–(4), which is for a possible application in heat conduction engineering by considering nonhomogeneous materials. The inverse problems are those in which one would like to determine the causes for an observed effect. The inverse problems are usually ill-posed. For the present inverse problem the observed effect is the temperature measurements in the rod at a final time. We are interesting to search the unknown coefficient $\alpha(x)$ in Eq. (1), which causes the effect we observe through measurement of temperature. For this inverse problem the measurement error may lead to a large discrepancy from the true cause.

In order to overcome this problem, there have been many studies, for example, Yeung and Lam [1], Keung and Zou [2], Lin et al. [3], Chang and Chang [4], Engl and Zou [5], Ben-yu and Zou [6], Jia and Wang [7], and references therein. Most of the studies applied the least squares method to estimate the heat conductivity in inverse heat conduction problems. Usually, the function form of the unknown heat conductivity is assumed, and the inverse problem is solved through an iteration process.

Ito and Kunisch [8,9] have proposed a very stable and efficient Lagrangian method for the identification of $\alpha(x)$ under a steady-state condition and with a smooth assumption on $\alpha(x)$. Then, Chen and Zou [10] extended that method to a non-smooth case in the steady-state elliptic system. Lam and Yeung [11] have employed a first-order finite difference method to determine the heat conductivity in a one-dimensional heat conduction equation. Yeung and Lam [1] extended that result by using a second-order finite difference technique. Chang and Chang [4] derived linear equations by using a finite volume method to determine the unknown heat conductivity without needing of iterations. For the problem governed by Eqs. (1)–(4), Liu et al. [12] have developed a highly accurate Lie-group estimation method, but required to know the boundary values of α a priori, and the measuring time t_f should be close to the initial time.

In practical applications we may encounter the inverse problem for composite materials or highly heterogeneous materials with a requirement to estimate discontinuous and periodically-oscillatory thermophysical parameters in a transient state. For this inverse problem, it is still a great challenge due to the lack of an efficient, accurate and stable method.

Present approach is based on the numerical method of line, which transforms PDEs into a system of ordinary differential equations (ODEs). Recently, Liu [13–15] has extended the group preserving scheme (GPS) developed previously by Liu [16] for ODEs to solve the boundary value problems (BVPs), and the numerical results reveal that the GPS is a rather promising method to effectively calculate the two-point BVPs. In the construction of the Lie group method for the calculations of BVPs, Liu [13] has introduced the idea of one-step GPS by utilizing the closure property of Lie group, and hence, the new shooting method has been named the Lie-group shooting method (LGSM).

On the other hand, in order to effectively solve the backward in time problems of parabolic PDEs, a past cone structure and a backward group preserving scheme have been successfully developed by the author, such that the one-step Lie-group numerical methods have been used to solve the backward in time Burgers equation by Liu [17], and the backward in time heat conduction equation by Liu et al. [18].

Liu [19–21] has used the concept of one-step GPS to develop numerical estimation methods for unknown temperature-dependent heat conductivity and heat capacity of a one-dimensional heat conduction equation. The Lie-group method possesses a great advantage than other numerical methods due to its group structure, and it is a powerful technique to solve the inverse problems of parameters identification.

The Lie-group method is originally used for the BVPs as designed by Liu [13–15] for direct problems. However, these methods are restricted only for two-dimensional ODEs, and here we will extend them to the multi-dimensional inverse problem. In a series of papers by the author and his coworkers, the Lie-group method reveals its excellent behavior on the numerical solutions of different type problems, for example, Chang et al. [22] to calculate the sideways heat conduction problem, Chang et al. [23] to treat the boundary layer equation in fluid mechanics, and Liu [24], Liu et al. [18], and Chang et al. [25,26] to treat the backward heat conduction equation, and Liu et al. [27] to treat the Burgers equation.

It should be stressed that the one-step Lie-group property is usually not shared by other numerical methods, because those methods do not belong to the Lie-group schemes. This important property as first pointed out by Liu [17] was employed to solve the backward in time Burgers equation. After that, Liu [19] has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended the Lie-group method to estimate the thermophysical properties of heat conductivity and heat capacity [12,20,21].

2. The numerical procedures

Eq. (1) can be written as

$$\frac{\partial T(x,t)}{\partial t} = \alpha'(x)\frac{\partial T(x,t)}{\partial x} + \alpha(x)\frac{\partial^2 T(x,t)}{\partial x^2} + h(x,t), \tag{5}$$

where $\alpha'(x)$ is the derivative of $\alpha(x)$ with respect to x. We adopt the numerical method of line to discretize the above derivatives with respect to x by

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=i\Delta x} = \frac{T_{i+1}(t) - T_{i-1}(t)}{2\Delta x}, \tag{6}$$

$$\left.\frac{\partial^2 T(x,t)}{\partial x^2}\right|_{x=i\Delta x} = \frac{T_{i+1}(t) - 2T_i(t) + T_{i-1}(t)}{(\Delta x)^2}, \tag{7}$$

where $\Delta x = \ell/(n+1)$ is a uniform spatial increment with the number n of interior grid points, and $x_i = i\Delta x$ are the discretized coordinates of x, at which the temperature is discretized as $T_i(t) = T(x_i, t)$. Here, $x_0 = 0$ and $x_{n+1} = \ell$. A similar finite difference can be used for $\alpha'(x)$.

In doing so, we can obtain a system of ODEs for T with t as an independent variable:

$$\begin{aligned}\dot{T}_i(t) = {} & \frac{\alpha_{i+1} - \alpha_{i-1}}{2\Delta x}\frac{T_{i+1}(t) - T_{i-1}(t)}{2\Delta x} \\ & + \alpha_i \frac{T_{i+1}(t) - 2T_i(t) + T_{i-1}(t)}{(\Delta x)^2} + h_i(t), \quad i = 1, \ldots, n,\end{aligned} \tag{8}$$

where $h_i(t) = h(x_i, t)$ and $\alpha_i = \alpha(x_i)$ are, respectively, the discretized quantities of $h(x,t)$ and $\alpha(x)$ at the nodal point x_i.

When $i = 1$, the term $T_0(t)$ appeared in Eq. (8) is determined by the first boundary condition in Eq. (2). Similarly, when $i = n$, the term $T_{n+1}(t)$ is determined by the second boundary condition in Eq. (2). Those are, $T_0(t) = F_0(t)$

and $T_{n+1}(t) = F_\ell(t)$. On the other hand, the terms α_0 and α_{n+1} are supposed to be measurable on boundaries.

The known initial condition is given by

$$T_i(0) = f(x_i), \quad i = 1, \dots, n, \tag{9}$$

which is obtained from Eq. (3) by a discretization. In summary, we have totally n ODEs in Eq. (8) to solve the $2n$ unknowns $T_i(t)$ and α_i, $i = 1, \dots, n$ with the aid of an extra n known values of

$$T_i(t_f) = F_m(x_i), \quad i = 1, \dots, n. \tag{10}$$

The Lie-group shooting method as first developed by Liu [13] will be extended and applied to Eq. (8). After giving a necessary mathematical background of the LGSM in next section, we will derive linear equations in Section 4 to determine the unknown coefficients α_i, $i = 1, \dots, n$.

3. Mathematical backgrounds

In order to explore our new method in a self-content fashion, let us first briefly sketch the group-preserving scheme (GPS) for ODEs and the one-step GPS in this section.

3.1. The GPS

Let us write Eq. (8) in a vector form:

$$\dot{\mathbf{T}} = \mathbf{f}(t, \mathbf{T}), \tag{11}$$

where

$$\mathbf{T} := \begin{bmatrix} T_1(t) \\ \vdots \\ T_n(t) \end{bmatrix},$$

$$\mathbf{f} := \begin{bmatrix} \frac{\alpha_2-\alpha_0}{2\Delta x}\frac{T_2-T_0}{2\Delta x} + \alpha_1 \frac{T_2-2T_1+T_0}{(\Delta x)^2} + h_1 \\ \frac{\alpha_3-\alpha_1}{2\Delta x}\frac{T_3-T_1}{2\Delta x} + \alpha_2 \frac{T_3-2T_2+T_1}{(\Delta x)^2} + h_2 \\ \vdots \\ \frac{\alpha_n-\alpha_{n-2}}{2\Delta x}\frac{T_n-T_{n-2}}{2\Delta x} + \alpha_{n-1} \frac{T_n-2T_{n-1}+T_{n-2}}{(\Delta x)^2} + h_{n-1} \\ \frac{\alpha_{n+1}-\alpha_{n-1}}{2\Delta x}\frac{T_{n+1}-T_{n-1}}{2\Delta x} + \alpha_n \frac{T_{n+1}-2T_n+T_{n-1}}{(\Delta x)^2} + h_n \end{bmatrix}. \tag{12}$$

$\mathbf{T}$ represents a vector form of the discretized temperatures at interior grid points, and the components of $\mathbf{f}$ represent the right-hand sides of Eq. (8). The dependence of $\mathbf{f}$ on t is due to the dependence of boundary condition (2) as well as the source functions h_i on t.

When both the vector $\mathbf{T}$ and its magnitude $\|\mathbf{T}\| := \sqrt{\mathbf{T}^t\mathbf{T}} = \sqrt{\mathbf{T}\cdot\mathbf{T}}$ are combined into a single augmented vector

$$\mathbf{X} = \begin{bmatrix} \mathbf{T} \\ \|\mathbf{T}\| \end{bmatrix}, \tag{13}$$

Liu [16] has transformed Eq. (11) into an augmented differential equations system:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \tag{14}$$

where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(t,\mathbf{T})}{\|\mathbf{T}\|} \\ \frac{\mathbf{f}^t(t,\mathbf{T})}{\|\mathbf{T}\|} & 0 \end{bmatrix} \tag{15}$$

is an element of the Lie algebra $so(n, 1)$ satisfying

$$\mathbf{A}^t\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{16}$$

and

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{17}$$

is a Minkowski metric. Here, $\mathbf{I}_n$ is the identity matrix, and the superscript t stands for the transpose.

The augmented variable $\mathbf{X}$ can be viewed as a point in the Minkowski space $\mathbb{M}^{n+1}$, satisfying the cone condition:

$$\mathbf{X}^t\mathbf{g}\mathbf{X} = \mathbf{T}\cdot\mathbf{T} - \|\mathbf{T}\|^2 = 0. \tag{18}$$

Then, Liu [16] developed a group preserving scheme (GPS) to guarantee that each $\mathbf{X}_k$ locates on the cone:

$$\mathbf{X}_{k+1} = \mathbf{G}(k)\mathbf{X}_k, \tag{19}$$

where $\mathbf{X}_k$ denotes the numerical value of $\mathbf{X}$ at the discrete time t_k, and $\mathbf{G}(k) \in SO_o(n, 1)$ satisfies

$$\mathbf{G}^t\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{20}$$

$$\det \mathbf{G} = 1, \tag{21}$$

$$G_0^0 > 0, \tag{22}$$

where G_0^0 is the 00-th component of $\mathbf{G}$.

3.2. One-step GPS

Throughout this paper we use the superscripted symbols $\mathbf{T}^0$ to denote the value of $\mathbf{T}$ at $t = 0$, and $\mathbf{T}^f$ the value of $\mathbf{T}$ at $t = t_f$.

Applying scheme (19) on Eq. (14) with a specified initial condition $\mathbf{X}(0) = \mathbf{X}^0$ we can compute the solution $\mathbf{X}(t)$ by the GPS. Assuming that the time stepsize used in the GPS is $\Delta t = t_f/K$, and starting from an augmented initial condition $\mathbf{X}^0 = ((\mathbf{T}^0)^t, \|\mathbf{T}^0\|)^t \neq \mathbf{0}$, we will calculate $\mathbf{X}^f = ((\mathbf{T}^f)^t, \|\mathbf{T}^f\|)^t$ at a final time $t = t_f$.

By applying Eq. (19) step-by-step we can obtain

$$\mathbf{X}^f = \mathbf{G}_K(\Delta t)\cdots\mathbf{G}_1(\Delta t)\mathbf{X}^0. \tag{23}$$

However, let us recall that each $\mathbf{G}_i$, $i = 1, \dots, K$, is an element of the Lie group $SO_o(n, 1)$, and by the closure property of Lie group, $\mathbf{G}_K(\Delta t)\cdots\mathbf{G}_1(\Delta t)$ is also a Lie group denoted by $\mathbf{G}$. Hence, from Eq. (23) it follows that

$$\mathbf{X}^f = \mathbf{G}\mathbf{X}^0. \tag{24}$$

This is a one-step Lie-group transformation from $\mathbf{X}^0$ to $\mathbf{X}^f$.

The remaining problem is how to calculate $\mathbf{G}$. While an exact solution of $\mathbf{G}$ is impossible, we can calculate an appropriate $\mathbf{G}$ through a numerical method by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie group generated from $\mathbf{A} \in so(n,1)$ by an exponential mapping is

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^t & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^t}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{25}$$

where

$$\widehat{\mathbf{T}} = r\mathbf{T}^0 + (1-r)\mathbf{T}^f, \tag{26}$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{t}, \widehat{\mathbf{T}}), \tag{27}$$

$$a = \cosh\left(\frac{t_f\|\hat{\mathbf{f}}\|}{\|\widehat{\mathbf{T}}\|}\right), \quad b = \sinh\left(\frac{t_f\|\hat{\mathbf{f}}\|}{\|\widehat{\mathbf{T}}\|}\right). \tag{28}$$

Here, we use the initial $\mathbf{T}^0$ and the final $\mathbf{T}^f$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $r \in [0,1]$ is a parameter to be determined and $\hat{t} = (1-r)t_f$. To stress its dependence on r we denote this $\mathbf{G}$ by $\mathbf{G}(r)$.

3.3. A universal one-step GPS

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\widehat{\mathbf{T}}\|}, \tag{29}$$

such that Eqs. (25) and (28) can be also expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{30}$$

$$a = \cosh(t_f\|\mathbf{F}\|), \quad b = \sinh(t_f\|\mathbf{F}\|). \tag{31}$$

From Eqs. (13), (24) and (31) it follows that

$$\mathbf{T}^f = \mathbf{T}^0 + \eta\mathbf{F}, \tag{32}$$

$$\|\mathbf{T}^f\| = a\|\mathbf{T}^0\| + b\frac{\mathbf{F}\cdot\mathbf{T}^0}{\|\mathbf{F}\|}, \tag{33}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{T}^0 + b\|\mathbf{T}^0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{34}$$

Eq. (32) is written as

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{T}^f - \mathbf{T}^0), \tag{35}$$

which being substituted into Eq. (33) and dividing both the sides by $\|\mathbf{T}^0\|$, we obtain

$$\frac{\|\mathbf{T}^f\|}{\|\mathbf{T}^0\|} = a + b\frac{(\mathbf{T}^f - \mathbf{T}^0)\cdot\mathbf{T}^0}{\|\mathbf{T}^f - \mathbf{T}^0\|\|\mathbf{T}^0\|}. \tag{36}$$

After inserting Eq. (35) for $\mathbf{F}$ into Eq. (31), a and b are now written as

$$a = \cosh\left(\frac{t_f\|\mathbf{T}^f - \mathbf{T}^0\|}{\eta}\right), \quad b = \sinh\left(\frac{t_f\|\mathbf{T}^f - \mathbf{T}^0\|}{\eta}\right). \tag{37}$$

Let

$$\cos\theta := \frac{(\mathbf{T}^f - \mathbf{T}^0)\cdot\mathbf{T}^0}{\|\mathbf{T}^f - \mathbf{T}^0\|\|\mathbf{T}^0\|}, \tag{38}$$

$$S := t_f\|\mathbf{T}^f - \mathbf{T}^0\|, \tag{39}$$

where $0 \leqslant \theta \leqslant \pi$ is the intersection angle between vectors $\mathbf{T}^f - \mathbf{T}^0$ and $\mathbf{T}^0$, and thus from Eq. (36) and (37) it follows that

$$\frac{\|\mathbf{T}^f\|}{\|\mathbf{T}^0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos\theta\sinh\left(\frac{S}{\eta}\right). \tag{40}$$

Upon defining

$$Z := \exp\left(\frac{S}{\eta}\right), \tag{41}$$

from Eq. (40) we obtain a quadratic equation for Z:

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{T}^f\|}{\|\mathbf{T}^0\|}Z + 1 - \cos\theta = 0. \tag{42}$$

Because the following condition is satisfied:

$$\left(\frac{\|\mathbf{T}^f\|}{\|\mathbf{T}^0\|}\right)^2 - 1 + \cos^2\theta \geqslant 0, \tag{43}$$

the solutions of Z are found to be

$$Z = \left(\frac{\|\mathbf{T}^f\|}{\|\mathbf{T}^0\|}\right)^{\pm 1} \quad \text{if } \cos\theta = \pm 1, \tag{44}$$

$$Z = \frac{\frac{\|\mathbf{T}^f\|}{\|\mathbf{T}^0\|} \pm \sqrt{\left(\frac{\|\mathbf{T}^f\|}{\|\mathbf{T}^0\|}\right)^2 - 1 + \cos^2\theta}}{1+\cos\theta} \quad \text{if } 0 \leqslant \pm\cos\theta < 1. \tag{45}$$

From Eqs. (41) and (39) it follows that

$$\eta = \frac{t_f\|\mathbf{T}^f - \mathbf{T}^0\|}{\ln Z}. \tag{46}$$

Therefore, we come to an important result that between any two points $(\mathbf{T}^0, \|\mathbf{T}^0\|)$ and $(\mathbf{T}^f, \|\mathbf{T}^f\|)$ on the cone, there exists a Lie-group element $\mathbf{G}(t_f) \in SO_o(n,1)$ mapping $(\mathbf{T}^0, \|\mathbf{T}^0\|)$ onto $(\mathbf{T}^f, \|\mathbf{T}^f\|)$, which is given by

$$\begin{bmatrix} \mathbf{T}^f \\ \|\mathbf{T}^f\| \end{bmatrix} = \mathbf{G}(t_f)\begin{bmatrix} \mathbf{T}^0 \\ \|\mathbf{T}^0\| \end{bmatrix}, \tag{47}$$

where $\mathbf{G}(t_f)$ is uniquely determined by $\mathbf{T}^0$ and $\mathbf{T}^f$ through the following equations:

$$\mathbf{G}(t_f) = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{48}$$

$$a = \cosh(t_f\|\mathbf{F}\|), \quad b = \sinh(t_f\|\mathbf{F}\|), \tag{49}$$

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{T}^f - \mathbf{T}^0) = \frac{\ln Z}{t_f}\frac{\mathbf{T}^f - \mathbf{T}^0}{\|\mathbf{T}^f - \mathbf{T}^0\|}. \tag{50}$$

In view of Eqs. (44), (45) and (38), it can be seen that $\mathbf{G}$ is fully determined by $\mathbf{T}^0$ and $\mathbf{T}^f$, and is independent on the vector field $\mathbf{f}$ in Eq. (11).

Notice that the above $\mathbf{G}$ is different from the one in Eq. (25). In order to stress its property as a Lie-group mapping between the quantities spanned a whole time interval $[0, t_f]$ we write it to be $\mathbf{G}(t_f)$. Conversely, $\mathbf{G}(r)$ is a function of r. However, these two Lie group elements $\mathbf{G}(r)$ and $\mathbf{G}(t_f)$ are both indispensable in our development of the Lie-group shooting method in the next section for the inverse problem of parameter identification.

The two Lie-groups $\mathbf{G}(r)$ and $\mathbf{G}(t_f)$ are constructed by different manners. When the former is obtained by using the generalized mid-point rule, the latter is a universal mapping between $(\mathbf{T}^0, \|\mathbf{T}^0\|)$ and $(\mathbf{T}^f, \|\mathbf{T}^f\|)$ independent on the vector field $\mathbf{f}$, which means that such a mapping is applicable to all ODEs systems. It is interesting that by letting $\mathbf{G}(r) = \mathbf{G}(t_f)$ we can derive the required governing equation below. From this point of view we may call our method the Lie-group shooting method (LGSM).

4. The Lie-group shooting method

4.1. $\mathbf{G}(r) = \mathbf{G}(t_f)$

Letting $\mathbf{G}(r) = \mathbf{G}(t_f)$ is essentially identical to letting the two $\mathbf{F}$'s in Eqs. (29) and (50) be equal, which leads to

$$\mathbf{T}^f = \mathbf{T}^0 + \frac{\eta}{\|\widehat{\mathbf{T}}\|}\hat{\mathbf{f}}, \tag{51}$$

where

$$\|\widehat{\mathbf{T}}\| = \|r\mathbf{T}^0 + (1-r)\mathbf{T}^f\|. \tag{52}$$

Up to here we have constructed a Lie-group shooting Eq. (51), which is a universal algebraic equation applicable to any vector field $\mathbf{f}$, and we may call it a natural field equation. This equation involves four quantities of $\mathbf{T}^0$, $\mathbf{T}^f$, $\mathbf{f}$ and r, the last of which is a single parameter uniquely determined by matching the target Eq. (4), i.e., $T(x, t_f) = F_m(x)$.

For the later use we write $\hat{\mathbf{f}}$ explicitly,

$$\hat{\mathbf{f}} = \begin{bmatrix} \frac{\alpha_2-\alpha_0}{2\Delta x}\frac{\hat{T}_2-\hat{T}_0}{2\Delta x} + \alpha_1\frac{\hat{T}_2-2\hat{T}_1+\hat{T}_0}{(\Delta x)^2} + \hat{h}_1 \\ \frac{\alpha_3-\alpha_1}{2\Delta x}\frac{\hat{T}_3-\hat{T}_1}{2\Delta x} + \alpha_2\frac{\hat{T}_3-2\hat{T}_2+\hat{T}_1}{(\Delta x)^2} + \hat{h}_2 \\ \vdots \\ \frac{\alpha_n-\alpha_{n-2}}{2\Delta x}\frac{\hat{T}_n-\hat{T}_{n-2}}{2\Delta x} + \alpha_{n-1}\frac{\hat{T}_n-2\hat{T}_{n-1}+\hat{T}_{n-2}}{(\Delta x)^2} + \hat{h}_{n-1} \\ \frac{\alpha_{n+1}-\alpha_{n-1}}{2\Delta x}\frac{\hat{T}_{n+1}-\hat{T}_{n-1}}{2\Delta x} + \alpha_n\frac{\hat{T}_{n+1}-2\hat{T}_n+\hat{T}_{n-1}}{(\Delta x)^2} + \hat{h}_n \end{bmatrix}, \tag{53}$$

where $\hat{T}_i = rT_i^0 + (1-r)T_i^f = rf(x_i) + (1-r)F_m(x_i)$, $\hat{h}_i = h_i(\hat{t})$, $i = 1, \ldots, n$, and $\hat{T}_0 = F_0(\hat{t})$ and $\hat{T}_{n+1} = F_\ell(\hat{t})$.

We should stress that $\hat{\mathbf{f}}$ is an unknown vector due to the appearance of the unknown coefficients α_i, but the vectors $\mathbf{T}^0$ and $\mathbf{T}^f$ are known, given by

$$\mathbf{T}^0 = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}, \quad \mathbf{T}^f := \begin{bmatrix} F_m(x_1) \\ \vdots \\ F_m(x_n) \end{bmatrix}. \tag{54}$$

Although $\hat{\mathbf{f}}$ is unknown we can evaluate it as follows. By using Eq. (51) we can solve $\hat{\mathbf{f}}$ by

$$\hat{\mathbf{f}} = \frac{\|\widehat{\mathbf{T}}\|}{\eta}(\mathbf{T}^f - \mathbf{T}^0). \tag{55}$$

Because $\mathbf{T}^0$, $\mathbf{T}^f$ and η calculated by Eq. (46) are all available, for a specified r, we can use Eq. (55) to calculate $\hat{\mathbf{f}}$, and then by Eq. (53) we can calculate α_i as follows. Let

$$a_i = \frac{\widehat{T}_{i+1} - \widehat{T}_{i-1}}{4(\Delta x)^2}, \quad i = 1, \ldots, n, \tag{56}$$

$$b_i = \frac{\widehat{T}_{i+1} - 2\widehat{T}_i + \widehat{T}_{i-1}}{(\Delta x)^2}, \quad i = 1, \ldots, n, \tag{57}$$

$$c_i = \hat{f}_i - \hat{h}_i, \quad i = 1, \ldots, n, \tag{58}$$

where $\hat{f}_i$ denotes the ith component of $\hat{\mathbf{f}}$, and from Eq. (53) we can obtain a linear equations system for α_i:

$$\begin{bmatrix} b_1 & a_1 & 0 & 0 & \cdots & 0 \\ -a_2 & b_2 & a_2 & 0 & \cdots & 0 \\ 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -a_{n-1} & b_{n-1} & a_{n-1} \\ 0 & \cdots & 0 & 0 & -a_n & b_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} c_1 + a_1\alpha_0 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n - a_n\alpha_{n+1} \end{bmatrix}. \tag{59}$$

We denote the above equation by

$$\mathbf{B}\boldsymbol{\alpha} = \mathbf{c}, \tag{60}$$

and consider the normal equation:

$$\mathbf{D}\boldsymbol{\alpha} = \mathbf{d}, \tag{61}$$

where

$$\mathbf{D} := \mathbf{B}^t\mathbf{B}, \tag{62}$$

$$\mathbf{d} := \mathbf{B}^t\mathbf{c}. \tag{63}$$

Then, the conjugate gradient method is used to solve Eq. (61) to obtain α_i, $i = 1, \ldots, n$.

4.2. How to choose r

Now we come to an important problem that how to choose r? In the paper by Liu et al. [12], they simply use $r = 1$ and show that the numerical solutions of α can be very accurate when t_f is rather small. The method used there is named the Lie-group estimation method (LGEM), which is a special case of the present method by using $r = 1$. When t_f increases, the LGEM may reduce its accuracy because $r = 1$ may be not the best one.

Then we turn our attention to the choice of r, and use numerical examples to demonstrate this idea. Under the known initial condition of $\mathbf{T}^0$ and the coefficients α_i solved from Eq. (61), we can return to Eq. (8) and integrate it to obtain $\mathbf{T}(t_f)$. The above process can be done for all r in the interval of $r \in [0,1]$. Among these solutions we can pick up the best r by searching the smallest error of

$$\min_{r\in[0,1]} \sqrt{\|\mathbf{T}(t_f) - \mathbf{T}^f\|^2}, \tag{64}$$

such that the final time condition specified by Eq. (4) can be fulfilled as best as possible.

When the process terminates, inserting the best r and $\hat{f}_i$ into Eq. (61) we can estimate the nonhomogeneous coefficient $\alpha(x)$.

5. Numerical tests

5.1. Example 1

Let us use the following example to demonstrate the above process. This example is given by

$$\alpha(x) = (x-3)^2, \tag{65}$$

$$h(x,t) = -7(x-3)^2 e^{-t}. \tag{66}$$

Under the boundary conditions

$$T(0,t) = 9e^{-t}, \quad T(1,t) = 4e^{-t}, \tag{67}$$

and the initial condition

$$T(x,0) = (x-3)^2, \tag{68}$$

the exact solution is given by

$$T(x,t) = (x-3)^2 e^{-t}. \tag{69}$$

In this identification of $\alpha(x)$ we have fixed $\Delta x = 1/40$ and $t_f = 0.01$ and 0.1. Applying the LGSM by choosing the best r, the solutions of α_i are almost equal to the exact ones with the maximum relative errors $1.0405625 \times 10^{-13}$ when $t_f = 0.1$ and $1.7781618 \times 10^{-13}$ when $t_f = 0.01$ as shown in Fig. 1a by the solid lines. The above maximum relative errors are much smaller than the one 0.0025 obtained by Yeung and Lam [1], and are also better than the ones presented by Liu et al. [12] with two orders. The L_2-norm errors are also plotted in Fig. 1a by the dashed lines. Overall, larger t_f reduces the accuracy. The above results suggest us to use $r = 0$ instead of $r = 1$. Even under a large $t_f = 1$, the present LGSM is also workable, where the absolute error between exact solution and numerical solution is plotted in Fig. 1b. The accuracy is very high up to the order of 10^{-12}.

We have carried out numerical estimations to assess the accuracy of the proposed inverse method without considering noise. When the measurement of temperature data is contaminated by noise, we can simulate the noisy temperature data, $\bar{F}_m(x_i)$, $i = 1,\ldots,n$ by adding random errors on exact temperatures $F_m(x_i)$, $i = 1,\ldots,n$ by

$$\bar{F}_m(x_i) = F_m(x_i) + \sigma e_i, \quad i = 1,\ldots,n, \tag{70}$$

where σ is the standard deviation of measurement errors, assumed to be the same for all measurements, and e_i is a

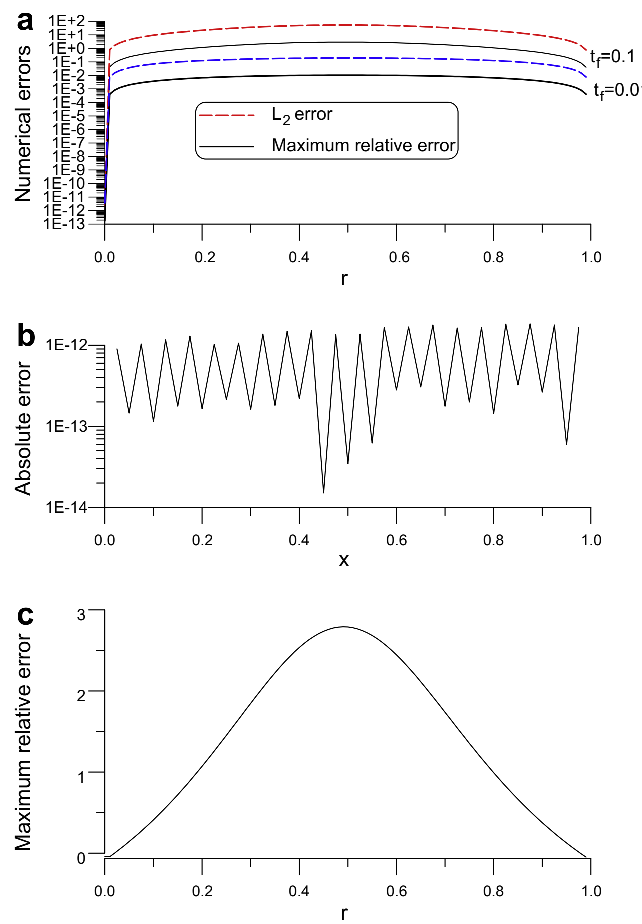

Fig. 1. For example 1: (a) comparing numerical errors with different final times with respect to r, (b) plotting absolute error, and (c) plotting maximum relative error with respect to r.

normally distributed random error. For normally distributed error, there is a 99% probability of the value of e_i lying in the range $-2.576 < e_i < 2.576$. The readers may refer Liu [21], where how to generate the normally distributed error was given.

In Fig. 1c the maximum relative error is plotted with respect to r under the same $\sigma = 0.01$. When $r = 0$ the maximum relative error is 0.021 and when $r = 1$ the maximum relative error is 0.017.

5.2. Example 2

Let us consider a one-dimensional heat conduction problem with

$$\alpha(x) = 1 + 0.25e^{-4(x-0.3)^2}, \tag{71}$$

$$h(x,t) = (x-0.6)^2(1-t)e^{-t} - \{2 + [0.5 - 4(x-0.3)(x-0.6)]e^{-4(x-0.3)^2}\}te^{-t}. \tag{72}$$

Under the boundary conditions

$$T(0,t) = 0.36te^{-t}, \quad T(1,t) = 0.16te^{-t}, \tag{73}$$

and the initial condition

$$T(x, 0.19) = 0.19e^{-0.19}(x - 0.6)^2, \tag{74}$$

the exact solution is given by

$$T(x, t) = (x - 0.6)^2 te^{-t}. \tag{75}$$

The one-dimensional domain $[0, 1]$ is discretized by $n + 2$ points including two end points, at which the two boundary conditions $T_0(t) = 0.36te^{-t}$ and $T_{n+1}(t) = 0.16te^{-t}$ are imposed on the totally n differential equations obtained from Eq. (8). In this identification of $\alpha(x)$ we have fixed $\Delta x = 1/40$, i.e., $n = 39$, and $t_f = 0.2$. Applying the LGSM, the numerical solutions of α_i by using $r = 1$ are almost equal to the exact ones with the maximum relative error 4.3×10^{-11} as shown in Fig. 2. The above maximum relative error is much smaller than the one 0.0004 obtained by Yeung and Lam [1].

In order to discuss the effect of mesh size on the accuracy of the calculated heat conductivity, the spatial domain $0 < x < 1$ is divided into 10, 20, and 40 subintervals for the above two demonstrated examples. The maximum relative errors of the calculated results are shown in Table 1 for mesh sizes $\Delta x = 0.1$, 0.05 and 0.025. In our calculations of these cases $r = 1$ is fixed, and the initial time is fixed to be $t_i = 0$ and the final time is fixed to be $t_f = 0.2$ for Example 1, and the initial time is fixed to be $t_i = 0.199$ and the final time is fixed to be $t_f = 0.2$ for Example 2. The results show that the maximum error is increased with decreasing mesh size. This property is very interesting and important, because in the measurement of temperatures we usually need to use thermocouples, whose number is better to limit as small as possible.

Comparing with Yeung and Lam [1] by second-order finite difference approach and with Chang and Chang [4] by finite volume method, the maximum error of the present study are much smaller. In contrast to our method, those methods required more measuring data in order to get more accurate results. It demonstrates that the proposed method of LGSM is an accurate and efficient technique to inverse estimation of heat conductivity.

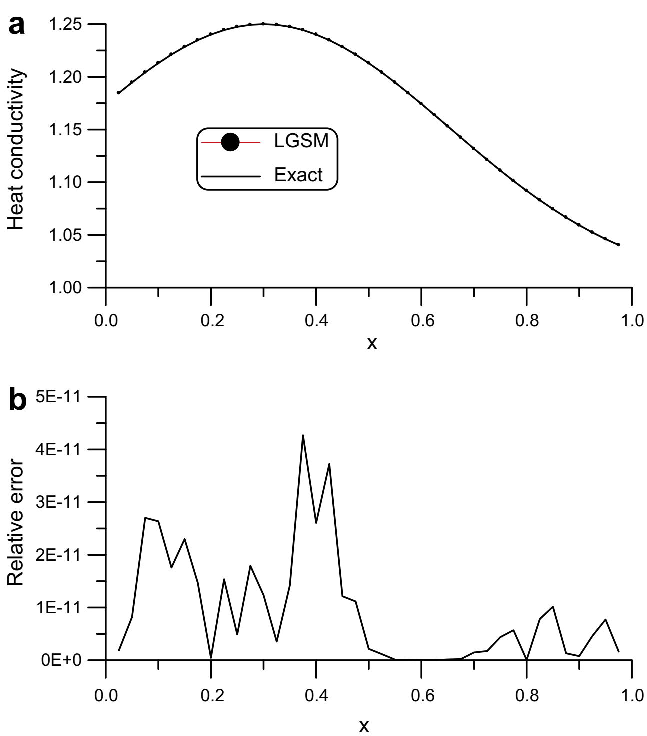

Fig. 2. For example 2: (a) comparing exact and LGSM numerical result of heat conductivity, and (b) plotting relative error.

6. Without knowing boundary values of α

6.1. A forward finite difference

The appearance of α_0 and α_{n+1} in Eq. (59) is due to that we have taken a central difference for α' in Eq. (5), and in the previous estimations we have assumed that α_0 and α_{n+1} can be measured. In this case we can obtain highly accurate results as shown by Examples 1 and 2 in Section 5. Now, we take a forward finite difference for both α' and $\partial T/\partial x$ in Eq. (5), and then we have

$$\dot{T}_i(t) = \frac{\alpha_{i+1} - \alpha_i}{\Delta x}\frac{T_{i+1}(t) - T_i(t)}{\Delta x} + \alpha_i \frac{T_{i+1}(t) - 2T_i(t) + T_{i-1}(t)}{(\Delta x)^2} + h_i(t), \quad i = 1, \dots, n. \tag{76}$$

Applying the same idea of LGSM on the above equation we can obtain a closed-form formula to estimate α_i:

$$\alpha_i = \frac{(\Delta x)^2}{\widehat{T}_i - \widehat{T}_{i-1}}\left[\frac{\widehat{T}_{i+1} - \widehat{T}_i}{(\Delta x)^2}\alpha_{i+1} + \hat{h}_i - \frac{\|\widehat{\mathbf{T}}\|}{\eta}(T_i^f - T_i^0)\right]. \tag{77}$$

The above equation can be sequentially used to find $\alpha_i, i = n, \dots, 1$ if we know α_{n+1} a priori. Here, α_{n+1} is the right-boundary value of α, which is now supposed to be an unknown value. This point is different from the previous paper by Liu et al. [12].

Table 1
Maximum errors on estimated heat conductivities for $t_f = 0.2$ comparing with other methods: Yeung and Lam [1] and Chang and Chang [4]

Example	$\Delta x = 0.1$			$\Delta x = 0.05$			$\Delta x = 0.025$		
	Present	[4]	[1]	Present	[4]	[1]	Present	[4]	[1]
1	2.4×10^{-14}	0.0292	0.0385	1.3×10^{-13}	0.0074	0.0099	1.9×10^{-13}	0.0019	0.0025
2	1.7×10^{-13}	0.0070	0.0073	5.2×10^{-13}	0.0017	0.0018	1.3×10^{-10}	0.00042	0.00045

For a specified r and assuming α_{n+1}, the above equation yields a sequence of α_i. Then we use the one-step GPS to calculate the data T_i^{f} by using these coefficients α_i. The final data is not immediately coincident with the measured data. However, we can change the value of α_{n+1} untill the resulting data T_i^{f} has a minimum discrepancy to the measured data. That is, we search the value of α_{n+1} by

$$\min_{\alpha_{n+1}>0} \sum_{i=1}^{n} |T_i^{\mathrm{f}} - F_m(x_i)|^2. \tag{78}$$

When the searching range of α_{n+1} is gradually refined we can obtain a very good approximation of the true boundary value of α. For example, we obtain $\alpha_{n+1} = 3.99966$ for Example 1, of which the true value is 4. In this calculation the parameters used are $r = 1$, $t_{\mathrm{f}} = 0.2$ and $\Delta x = 0.025$.

Fig. 3 displays the numerical results. It can be seen that these two curves as shown in Fig. 3a of numerical and exact ones are very close, and the error as shown in Fig. 3b is smaller than 8.2×10^{-5}. When we compare this result with those calculations by Yeung and Lam [1] and Chang and Chang [4], it is still better about two orders.

Even under a large noise with $\sigma = 0.02$ the maximum relative error is about in the order of 6.3×10^{-3}, of which the numerical result and numerical error are shown, respectively, in Figs. 3a and b. From this example it can be seen that our method is also applicable for the case under the noised disturbance on the measured data. The robustness of the present method is better than that of Chang and Chang [4].

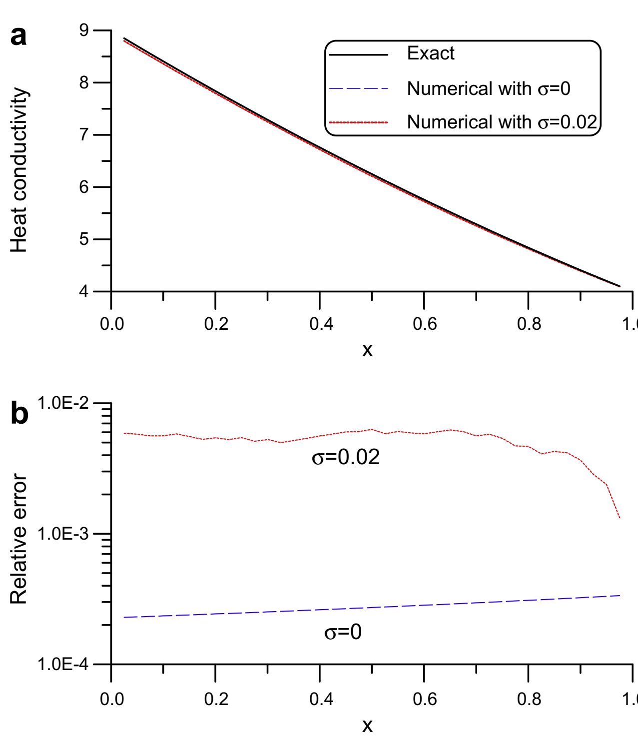

Fig. 3. For example 1: (a) comparing exact and numerical results under $\sigma = 0, 0.02$ of heat conductivity, and (b) plotting relative errors.

6.2. Example 3

This problem is under the following observed data:

$$F_m(x) = \sin \pi x \exp[\sin \pi t_{\mathrm{f}}], \tag{79}$$

which is obtained from

$$T(x,t) = \sin \pi x \exp[\sin \pi t]. \tag{80}$$

But the identified function $\alpha(x)$ is highly discontinuous and oscillatory given as follows:

$$\alpha(x) = \begin{cases} 2 & x \in [0, 0.3], \\ 4 & x \in (0.3, 0.6), \\ 2 + \sin(10\pi x) & x \in [0.6, 1]. \end{cases} \tag{81}$$

The function $h(x,t)$ is calculated as

$$h(x,t) = \begin{cases} \{\pi \cos \pi t + 2\pi^2\} \exp[\sin \pi t] \sin \pi x \\ \quad x \in [0, 0.3], \\ \{\pi \cos \pi t + 4\pi^2\} \exp[\sin \pi t] \sin \pi x \\ \quad x \in (0.3, 0.6), \\ \{\pi \cos \pi t + (2 + \sin 10\pi x)\pi^2\} \exp[\sin \pi t] \sin \pi x \\ -10\pi^2 \cos \pi x \cos 10\pi x \exp[\sin \pi t] \\ \quad x \in [0.6, 1]. \end{cases} \tag{82}$$

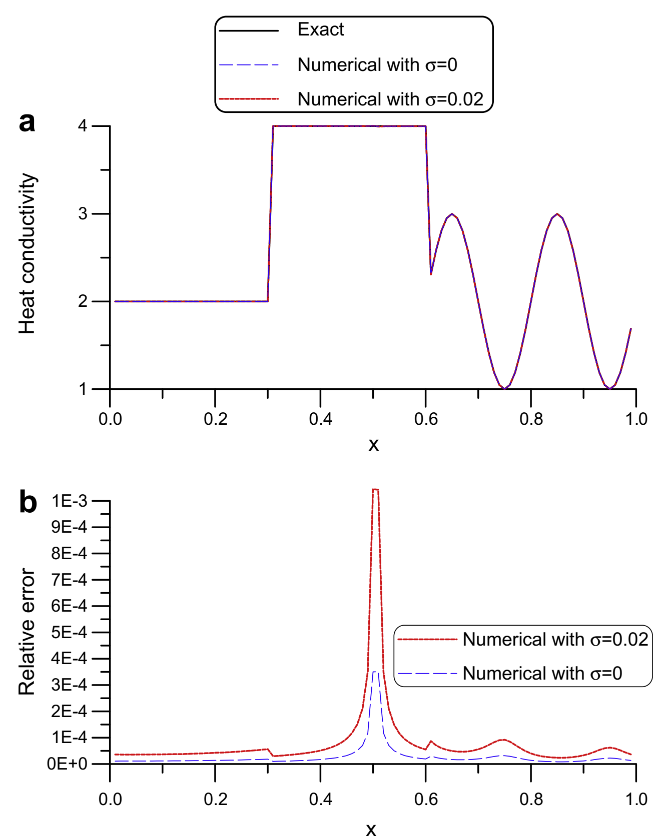

Fig. 4. For example 3: (a) comparing exact and numerical results under $\sigma = 0, 0.02$ of heat conductivity, and (b) plotting relative errors.

In this identification of $\alpha(x)$ we have fixed $\Delta x = 1/100$ and $t_f = 0.2$. Applying Eq. (77), the solutions of α_i are almost equal to the exact ones with the maximum relative error 3.5×10^{-4} as shown in Fig. 4. The error is smaller than the one 0.054 calculated by Keung and Zou [2]. Even under a large noise with $\sigma = 0.02$ the maximum relative error is small in the order of 10^{-3}, of which the numerical result and numerical error are shown, respectively, in Figs. 4a and b. From this example it can be seen that our method is also applicable to the estimation of highly discontinuous and oscillatory parameter. The robustness of the present method is better than that by Liu et al. [12].

Through these identifications of $\alpha(x)$ in Examples 1–3, it can be seen that our estimations are rather accurate, no matter the function $\alpha(x)$ is smooth or non-smooth. The accuracy and efficiency of our LGSM is much better than other methods.

6.3. A simple view of Eq. (1)

When a final time measurement of temperatures at t_f can be made as short as possible, the duration of time interval is very short, and we can rewrite Eq. (1) as an ODE at a time t_0 by letting

$$\frac{T(x,t_f)-T(x,0)}{2\Delta t} = \frac{\partial}{\partial x}\left[\alpha(x)\frac{\partial T(x,t_0)}{\partial x}\right] + h(x,t_0), \quad 0<x<\ell, \tag{83}$$

where $t_0 = t_f/2 = \Delta t$.

Integrating both the sides of Eq. (83) and leaving a constant C at there, we obtain

$$\int_0^x \frac{T(\xi,t_f)-T(\xi,0)}{2\Delta t}\,d\xi + C = \alpha(x)\frac{\partial T(x,t_0)}{\partial x} + \int_0^x h(\xi,t_0)\,d\xi, \tag{84}$$

where we can approximate $\partial T(x,t_0)/\partial x$ by

$$\frac{\partial T(x,t_0)}{\partial x} = \frac{1}{2}\left[\frac{\partial T(x,t_f)}{\partial x} + \frac{\partial T(x,0)}{\partial x}\right]. \tag{85}$$

Then we have

$$\alpha(x) = \frac{2}{\left[\frac{\partial T(x,t_f)}{\partial x} + \frac{\partial T(x,0)}{\partial x}\right]}\left[C + \int_0^x \frac{T(\xi,t_f)-T(\xi,0)}{2\Delta t}\,d\xi - \int_0^x h(\xi,t_0)\,d\xi\right]. \tag{86}$$

Here C is an unknown, which can be determined by inserting the above $\alpha(x)$ into Eq. (8) and then using a finer time stepsize, for example, $\Delta t/N$, and the fourth-order Runge–Kutta method to integrate it from $t=0$ to t_f to obtain $T(x_i,t_f)$, among which we can select the best C by taking the minimum of

$$\min_C \sum_{i=0}^{n+1} |T(x_i,t_f) - F_m(x_i)|^2. \tag{87}$$

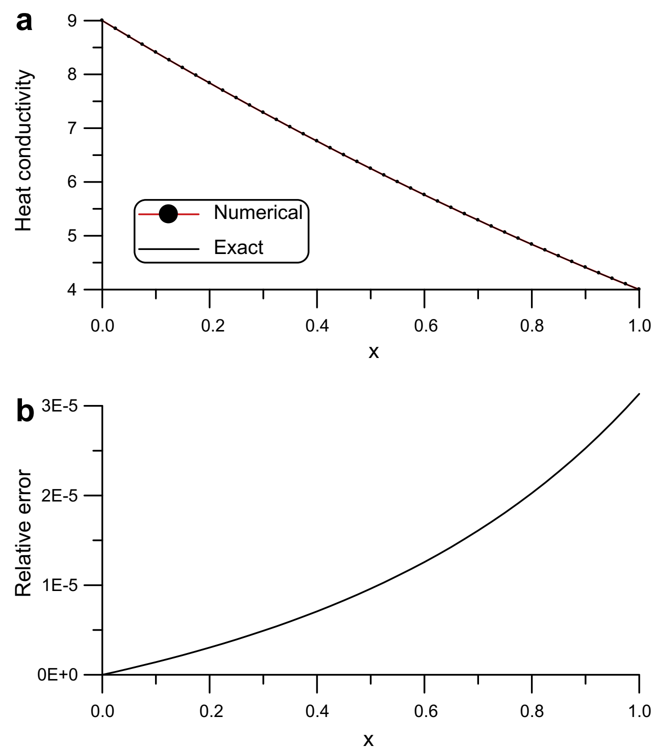

Fig. 5. For example 1: (a) comparing exact and a simple numerical solution of heat conductivity, and (b) plotting relative error.

In Fig. 5 we plot the numerical results for Example 1 by fixing $\Delta t = 0.005$, $t_f = 0.01$ and $N = 10$. It can be seen that these two curves as shown in Fig. 5a of numerical and exact ones are very close, and the error as shown in Fig. 5b is smaller than 3×10^{-5}.

7. Conclusions

The LGSM by using central or forward finite difference and a simple approach by a single ODE have been developed for the inverse estimation of spatially-dependent heat conductivity in a one-dimensional rod. A system of linear equations for the LGSM is constructed by using temperature data at initial and final times and heat generation at discrete points. The unknown heat conductivity can be solved explicitly in matrix form. The advantages of the present method are that no prior information about the functional form of heat conductivity is necessary, no initial guesses and no iterations are required, and the inverse solution can be efficiently solved in a linear domain. When the forward difference is employed in the LGSM, the estimation formula could be further written in a closed-form, which can be sequentially generating the correct heat conductivity coefficients. Its defense to the noised disturbance is remarkable. The accuracy and robustness of the present algorithms are confirmed by comparing the estimated results with exact solutions. It shows that a fairly accurate estimation can be achieved even under a large measurement error.

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